AUTO-CONSTRAINING SCORING FORMULAE FOR AIRCRAFT EFFICIENCY

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The History of CAFE Efficiency Formulae

Since it began measuring aircraft efficiency in 1981, CAFE discovered that certain conventional aerodynamic parameters could be defined using only the three key quantities above: **V** for Velocity, $\mathbf{W_p}$ for weight of payload, and **V/gph** (Velocity divided by gallons per hour) for miles per gallon.

The first aerodynamic parameter so defined is the equivalent flat plate drag area of the aircraft, "**F**". We know that an aircraft's **F** represents the number of square feet of frontal area of a flat plate whose drag is equivalent to the total parasite drag of that aircraft. In simplified terms, **F** can be represented as $\mathbf{gph/V^3}$. The term **F** is an inverse measure of the sleekness of the aircraft - i.e., its ability to fly at high speed using very little fuel. In terms of efficiency, the smaller the value of **F**, the more efficient the vehicle. Therefore, one wants to maximize the inverse of **F** , i.e. **1/F** , which can be represented as velocity cubed divided by gallons per hour of fuel flow ($\mathbf{V^3 / gph}$).

Because **gph** is necessarily dependent on engine size, the value of **F** obviously will vary in some proportion to the size of the engine and hence the size of the aircraft. **F** is known to be proportional to the square feet of projected frontal area of an aircraft, i.e., its dimension squared. $\mathbf{W_p}$, the weight of the aircraft's payload, relates to the cubic feet of volume available in the aircraft, i.e., its dimension cubed. In order to scale **F** according to the payload of the aircraft, we can multiply **1/F** x the square of the cube root of $\mathbf{W_p}$ as shown: $\mathbf{1/F \times (W_p^{1/3})^2}$. This then becomes $\mathbf{1/F \times W_p^{2/3}}$, or $\mathbf{(V^3 / gph) \times W_p^{2/3}}$, a term that we can call the ***scaled inverse flat plate drag area***. Note that with this term, velocity is of utmost importance because it appears as a cubed term in the equation. An aircraft's optimum value for its ***scaled inverse flat plate drag area*** occurs at its maximum velocity.

Another key aerodynamic parameter by which it is useful to rate efficiency is the lift to drag ratio or **L/D**. This can be represented in two ways; thermodynamically and aerodynamically:

1) The ***thermodynamic*** **L/D ratio** is represented as MPG x Cabin Payload or, alternatively, as $\mathbf{(V/gph) \times W_p}$. This parameter, like the scaled inverse flat plate drag area term mentioned above, entails offsetting the MPG, (because it is a term inversely dependent on size), by the weight of the payload carried, which is directly dependent on size. ***Thermodynamic*** **L/D ratio** emphasizes optimizing the engine's efficiency or brake specific fuel consumption, propulsive efficiency, structural efficiency, fuel energy density, etc. Note that with this term, MPG times payload, velocity is not the main issue, and a very high score could be obtained at a very low velocity.

2) The ***aerodynamic*** **L/D max**, represented by: $\mathbf{.8862 \times b_e/(gph/V^3)^{0.5}}$ is an alternative, aerodynamic version of the **L/D** parameter by which to score an aircraft's efficiency, and here the b_e stands for the effective wingspan. Because b_e is not one of the 3 fundamental efficiency quantities of interest, it is

helpful to replace it in the expression for ***aerodynamic*** **L/D max** with $\mathbf{W_p}$ raised to some exponent as a reasonably proportionate substitute term based on size. Since $\mathbf{W_p}$ scales with the aircraft's cubic feet of volume and wingspan scales with the single dimension of span, we can relate the wingspan to the payload as $\mathbf{b_e \sim W_p^{1/3}}$. This then allows the ***aerodynamic*** **L/D max** to be expressed as: $\mathbf{.8862 \times W_p^{1/3}/(gph/V^3)^{0.5}}$

We can now combine the three key aerodynamic parameters above to create sample scoring formulas with various emphasis.

The first such sample formula is to simply use $\mathbf{1/F \times W_p^{2/3}}$ which is $\mathbf{V^3 / gph \times W_p^{2/3}}$, the ***scaled inverse flat plate drag area***. By recognizing that **V/gph** is the same as **MPG**, it simplifies to:

$$\mathbf{V^2 \times MPG \times W_p^{1/3}} \quad \text{equation 1.0}$$

Equation 1.0 ignores the aircraft's lift to drag ratio and places very high emphasis on top speed. Note that the exponent for velocity is 2.0 and is twice as large as the 1.0 value of the exponent for MPG. This formula does not adequately emphasize fuel efficiency and instead makes for a competition that is a flat out speed race, much like the original Lowers Baker Falck 500 races at Oshkosh.

The second such sample formula is: multiply the inverse of **F** times the ***thermodynamic*** **L/D ratio** to obtain:

$$\mathbf{V^3 / gph \times [\ V/gph \times W_p\ ]}$$

or

$$\mathbf{V^4 / gph^2 \times W_p}$$

Taking the square root of this expression yields:

$$\mathbf{V \times MPG \times W_p^{0.5}} \quad \text{equation 2.0}$$

This expression was used in the 1981 CAFE 250 race as the flight efficiency formula. In retrospect, it was deemed to give inappropriate credit for payload. This implies that ***scaled inverse flat plate drag area*** should have been used in the formula in order to scale **1/F** appropriately. Doing so would have led to the formula:

$$\mathbf{V \times MPG \times W_p^{0.83}} \qquad \text{equation 2.1}$$

However, equation 2.0 and 2.1 were also found to promote competing aircraft to fly at only 50% power, an unrealistically low speed. A better formula was needed.

The third such sample formula is to multiply **Velocity x *thermodynamic* L/D ratio** to get **V x [V/gph) x W_p]** or:

$$V \times MPG \times W_p \quad \text{equation 3.0}$$

Equation 3.0 is the CAFE 400 formula of 1982. It combines a non-size-dependent quantity, velocity, with the scaled parameter of ***thermodynamic* L/D ratio**. Here, velocity, MPG and payload all have an exponent of 1.0. It was found in the CAFE 400 race results that this expression de-emphasizes velocity and gives disproportionate credit for aircraft payload. This CAFE 400 formula again optimized at a rather low velocity corresponding to about 50% power settings for most aircraft, depending upon their span loading. CAFE named this velocity "Carson Speed" in honor of Professor Bud Carson at the US Naval Academy, whose excellent paper, (**Carson, B.H.. Fuel Efficiency of Small Aircraft, AIAA J. of Aircraft, Vol. 19, No. 6, pp. 473–479, June 1982)** elucidated the aerodynamic derivation of its importance. Dr. Carson's paper describes Carson Speed as the 'least wasteful way of wasting' because it optimizes at the least "dollars per mile per hour". Interestingly, Carson Speed always turns out to be 32% faster than speed for best L/D, and can be found by drawing a line through the origin that is tangent to the drag polar curve.

The problem is that many people will gladly pay a few dollars more to go faster than Carson Speed, because they place high importance on saving time. CAFE considered the 50% power setting promoted by the CAFE 400 formula (equation 3.0) to be unrealistically low because aircraft normally cruise at 65-75% power. Therefore, CAFE empirically revised the formula to be:

$$V^{1.3} \times MPG \times Wp^{0.6} \quad \text{equation 4.0}$$

This new formula, equation 4.0,worked beautifully for the subsequent CAFE competitions for aircraft of from 2 to 8 seats. Some justification for the payload exponent of 0.6 and the velocity exponent of 1.3 is given below:

If we try a fifth sample formula that attempts to combine the **scaled inverse flat plate drag area** and ***aerodynamic* L/D max**, then we are effectively asking the aircraft to equally optimize each of these two opposing but non-scale related factors. The flat plate term optimizes at Vmax while the L/D term optimizes at V_y .

This formula would be written as:

$$[\ (V^3/gph) \times W_p^{2/3}\ ] \times [\ .8862 \times W_p^{1/3}/(gph/V^3)^{0.5}\ ]$$

Neglecting the constant, this can be simplified to:

$$[\ V^3/gph \times W_p^{2/3}\ ] \times [W_p^{1/3}/(gph^{0.5}/V^{1.5})\ ]$$ which can rearranged to:

$$(\ V^{4.5} \times W_p\ ) / gph^{1.5}$$ which, by taking its 2/3 root, simplifies to:

$\mathbf{V^3 \times W_p^{2/3} / gph}$ which can be written as:

$\mathbf{V^2 \times MPG \times W_p^{0.6667}}$ equation 5.0

Equation 5.0 affirms a fractional exponent of about 0.6 for payload. However, its exponent of 2.0 for Velocity would incentivize flying at a very high power settings near Vmax. Consequently, a search for the ideal exponent for Velocity was undertaken by CAFE, using an empirical method set forth below.

The Velocity for Best CAFE Score: V_{BC}

The empirical approach

If one wants to create a scoring formula to simultaneously optimize two or more aerodynamic parameters that inherently oppose one another from a design perspective, multiplying the parameters times one another gives them equal weight in determining the score. If one applies some exponent other than 1.0 to any of the parameters, one will accordingly increase or decrease that parameter's influence upon the result. One may choose to increase the weighting of a parameter that is a more important feature and deserves more emphasis, but such emphasis must be justified.

Consider the expression, $V^?$ /gph, where we divide a vehicles's Velocity (raised to some exponent) by its fuel flow in gallons per hour. We can set the exponent for velocity in this expression to place more or less emphasis on time as compared to fuel economy. Saving time, at higher Velocity, takes more fuel in gallons per hour. Different people place different values on time versus fuel. For aircraft, this is an important consideration. Aircraft fuel economy varies with aircraft velocity, and therefore the appropriate or desired velocity is not always top speed. The key speeds to consider are:

Vmax or maximum speed
$\mathbf{V_{BC}}$ the speed for Best CAFE score, people's usual choice in valuing time vs. fuel
$\mathbf{V_{Ca}}$ also know as Carson speed, 32% faster than $\mathbf{V_y}$
$\mathbf{V_y}$ or speed for best MPG or maximum Lift to Drag ratio (L/D), 32% faster than $\mathbf{V_x}$
$\mathbf{V_x}$ or speed for minimum power and thus for lowest possible fuel cost

We will see below that we can set the exponent for Velocity in the expression $V^?$ /gph to select or incentivize for each of these speeds. The exponents for velocity that we will examine are: 0.0, 1.0, 2.0, 2.3 and 3.0. See the graph below.

When we optimize $V^{0.0}$/gph, we find that speed does not matter, since the expression is really 1/gph. This is the exact situation at V_x ,the speed for minimum power. V_x is the speed that allows sustained flight at the lowest possible fuel flow in gph. Because V_x is also known as the max endurance speed, the only thing that really matters to the pilot flying at Vx is the value of gph. V_x is an unalterable speed characteristic of any particular aircraft design and is commonly called best angle of climb speed.

When we optimize $V^{1.0}$/gph, we find that the optimum speed occurs at V_y, which we know is 31% higher than V_x . Here, the formula equates to MPG and emphasizes MPG alone. This makes sense because V_y is an aircraft's characteristic, unalterable speed for max range or max MPG. It is also the speed for maximum L/D and thereby is the speed at which the aircraft has its best glide ratio. It is generally too slow a speed for normal cruise flight except for unusual missions like the Voyager World Flight.

When we optimize $V^{2.0}$/gph, we find that the optimum speed occurs at Carson speed, the speed for optimum V x MPG. By multiplying the two numbers together, the expression V x MPG simultaneously optimizes the trade-off balance between speed and MPG. Curiously, Carson speed or V_{Ca} , is 32% higher than V_y . We find that V_{Ca} is generally too slow a cruise speed to suit most pilots, equating to about 50% power in most aircraft.

When we optimize $V^{3.0}$/gph, we are optimizing a ratio that proportions horsepower with the cube root of velocity. We know that such a ratio represents the flat plate drag of the aircraft, which is lowest at the aircraft's maximum velocity, Vmax, where the induced drag is usually very small or negligible. This calls for 100% power and is generally too high a power setting for cruise flight.

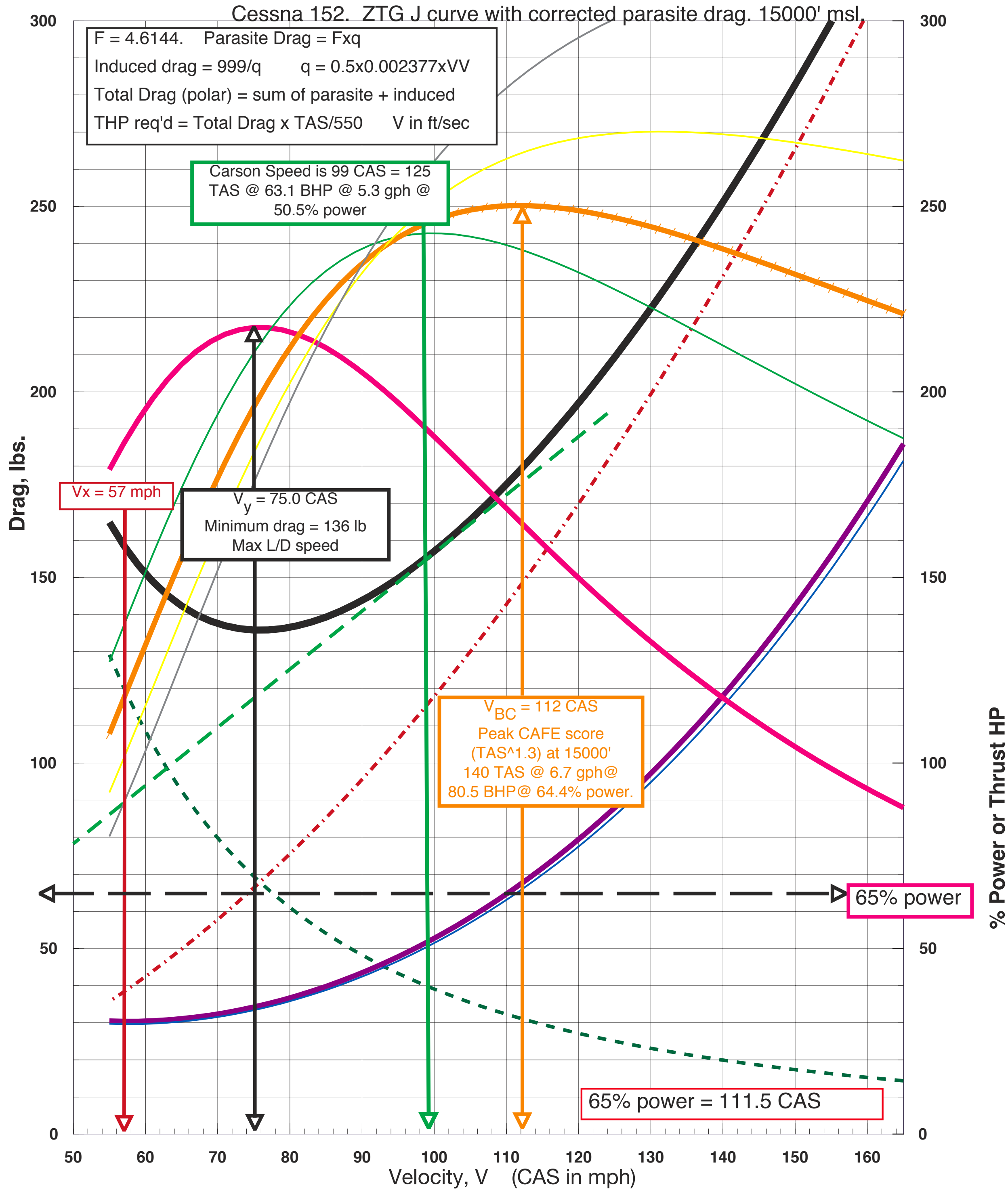
In the CAFE formula for aircraft efficiency, $V^{1.3}$ x MPG x $W_p^{0.6}$, we have the expression $V^{2.3}$/gph whose optimum speed falls between Carson speed and Vmax. Is this exponent of 2.3 for velocity one based solely on the empirical results of the CAFE 400 air races or does it have some valid physical formulaic justification?

On the generic "J" curve drag polar plot of aircraft Velocity versus drag in pounds, (see graph above) V_y is the point at the bottom of the J, where parasite drag is 50% of the total and induced drag is 50% of the total. From V_y , one can either go 31% slower to V_x , or 31% faster to V_{Ca} . Going slower means increasing the proportion of induced drag to parasite drag while going faster means increasing the proportion of parasite drag to induced drag. Going even faster than Carson Speed means further increases in parasite drag.

The x-y graph below correlates various CAFE type scoring formulae with a classic drag polar as well as with level flight cruise fuel flow settings.

The findings clearly show how the exponent applied to the Velocity factor in the equation determines the choice of optimum cruise speed and fuel mixture setting. It became our contention at the CAFE Foundation that in the overall design of propeller-driven aircraft, an efficiency formula using an exponent of 1.3 for Velocity serves very well. It appears to optimize the power setting to about 65% power and drives the mixture setting to be lean of peak or stoichiometric EGT but not excessively so. In addition, the exponent of 1.3 for Velocity can point to the ideal size of engine for any aircraft of known drag polar. For these reasons, the CAFE Challenge Trophy is awarded on the basis of a 1.3 exponent for Velocity.

CAFE FOUNDATION



The exponent for Velocity in the CAFE formula determines the %power at which the score optimizes. Assumes 82% prop efficiency, 15000' density altitude with sigma=0.63 for HPand 0.5 bsfc. CAFE score = V^1.3 x MPG. Ideal cruise power is 65%.

Parasite — Carson's Speed — V^0 x MPG — V^1.6 x MPG
Induced — THP Req'd 15K — V^1.0 x MPG — V^1.9 x MPG
Drag Polar — % power — V^1.3 x MPG

The V_{BC} concept, i.e., Velocity for Best CAFE score, occurs at the maximum of Velocity to the 1.3 power times MPG and is an excellent standard by which a pilot can determine the choice of cross-country ideal cruise speed and compare various aircraft to one another.

APPENDIX:

Professor B. H. Carson's Notes on Flat Plate Drag: a transcribed handwritten note, 1/28/1982:

Note on the Determination of Equivalent Parasite Area from Flight Test Data

It is frequently assumed that the power required for a propeller airplane scales as the cube of the velocity.

This is because: P = D*V = Cd*(1/2_V^2)V = Cd*1/2_V^3S

Where all symbols are conventionally defined. Calling Cd*S = F, the equivalent parasite area, then

F = 2P/_V^3 and hence, in comparing power and airspeed, it follows that

$P_2/P_1 = (V_2/V_1)^3$ equation (1)

Where of course the comparison is made at the same altitude.

The purpose of this brief note is to show that equation (1) is incorrect except at airspeeds near the maximum, and that the value of "F" thus predicted by the preceding equation is generally in great error.

To begin, we note that the expression for specific drag (D/W) can be written:

$D/W = AV^2 + B/V^2$ where

A =_F/2W and B = 2W/(_πb^2e) thus, the specific power, P_s, (=P/W) is

$P_s = AV^3 + B/V$ equation (2)

This expression has a minimum in V (= V_1)
Given by:

$V_1 = (B/3A)^{0.25}$ equation (3)

Which yields:

$P_{s1} = (4/3^{0.75})(B^3A)^{0.25}$ equation (4)

i.e., the minimum specific power. Then, after some algebraic reduction, equation (2) becomes

$$P_s/P_{s1} = 1/4 * (V/V_1)^3 + 3/4 * (V_1/V) \qquad \text{equation (5)}$$

Now suppose it takes 100 HP to fly a certain aircraft at 80 mph, corresponding to minimum power. Then, to fly twice as fast, at 160 mph, equation (1) predicts 800 HP required, whereas equation (5) predicts:

$$P = 100[(1/4)(2)^3 + 3/4(1/2)] = 238 \text{ HP}$$

Thus, equation (1) is inherently in error. Specifically, the error lies in neglecting the variation in Cd with airspeed.

These results can be directly applied to the determination of "F". First, we note that V_1 is related to VmaxL/D by the expression:

$$V_1 = \text{VmaxL/D} / 3^{0.25} = 0.76 \text{ VmaxL/D}$$

and, as it happens, VmaxL/D happens to coincide with the best rate of climb speed for many aircraft. Knowing VmaxL/D establishes V_1 to a close approximation and it is Vx. At this point, an example will prove helpful. Suppose the following data are available for a certain aircraft:

W = 3000 lb S = 180 sq ft b = 30 ft e (est) = 0.8
VmaxL/D = 110 mph EAS eta = 85% BHP = 285 (installed)

Then, Vx = 0.76 * 110 = 83.6 mph = 122.6 ft/sec.

From equation (3),

$$B/3A = (122.6)^4 = 2.26 * 10^8 \text{ and thus } A = [1.475 * 10^{-9}] *B$$

From the above,

$$B = (2*3000)/(0.002378)*(3.1416)*(30^2)(0.78) = 1.144 * 10^3 \ (\text{ft}^2/\text{sec}^2)$$

And hence,

$$A = (1.475 * 10^{-9})*(1.144 * 10^3) = (1.686 * 10^{-6}) \ (\text{sec}^2/\text{ft}^2)$$
$$= (\_* F)/2W$$

So therefore:

$$F = 1.686 * 10^{-6}) * (2 * 3000/0.002378) = 4.25 \text{ sq ft.}$$

At the same time, from equation (4):

$$P_{s1} = (4/3^{0.75}) * (B^3 * A)^{0.25} = 12.45 \text{ ft-lb/lb}$$

x 3000 lb /(550 ft-lb/hp) = 68 HP

The manufacturer of the Bonanza claims that the aircraft cruises at 165 mph @ 65% power. We can check this from equation (5), i.e.,

$$P \text{ (req'd, 165 mph)} = 68\,[1/4((165/83.6)^3) + 3/4(83.6/165)] = 157 \text{ HP req'd}$$

Since BHP = THP/eta then

$$BHP = 157/0.85 = 184 \text{ BHP and } [184/285] * 100 = 65\%$$

Hence the published data agree with calculations.
Note that if the first equation were used with the 65% data, then

$$F = (2 * 184 * 550)/(0.002378) * (1.467 * 165)^3 = 6.00 \text{ sq ft}$$

which is 40% in error.